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On E-discretization of tori of compact simple Lie groups

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Abstract
Three types of numerical data are provided for compact simple Lie groups G of classical types and of any rank. These data are indispensable for Fourier-like expansions of multidimensional digital data into finite series of E-functions on the fundamental domain F^e. Firstly, we determine the number $\left|F^e_M\right|$ of points in F^e from the lattice $P^\vee_M$, which is the refinement of the dual weight lattice $P^\vee$ of G by a positive integer M. Secondly, we find the lowest set Λ^e_M of the weights, specifying the maximal set of E-functions that are pairwise orthogonal on the point set F^e_M. Finally, we describe an efficient algorithm for finding the number of conjugate points to every point of F^e_M. Discrete E-transform, together with its continuous interpolation, is presented in full generality.

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1. Introduction

The E-discretization of this paper differs in an important way, both theoretically and practically, from the ‘ordinary’ discretization studied in [1]. First, we point out what the two approaches have in common, then we underline their differences.

Both discretizations must have an underlying compact simple or semisimple Lie group G of any rank $n < \infty$ and of any type. The rank is equal to the number of variables involved in the process. Then we introduce new classes of multivariate special functions, the C- and S-functions in [1], and the E-functions (44) here. The functions are orthogonal on a finite region F of the n-dimensional real Euclidean space $\mathbb{R}^n$, as continuous functions as well as functions restricted to a fragment of a lattice $L \cap F \subset \mathbb{R}^n$. The lattice can have any density but its symmetry is imposed by the underlying Lie group. Thus, in $\mathbb{R}^n$ there are as many different lattices and special functions orthogonal on them as there are semisimple Lie groups of rank

n. The families of C-, S- and E-functions were recognized and named in [2]. They generalize the common cosine, sine and exponential functions of one variable. Many of their properties are described in reviews [3–5].

The orbit functions C and S are built using the finite reflection group W, attached to each G, and called the Weyl group of G. The E-functions are built using the even subgroup $W^e \subset W$ which is not a reflection group. Much less specific information is available about such groups in the literature. The E-functions are simpler, for the same G, than the orbit functions of type C or S. More precisely, an orbit function is a sum/difference of two E-functions. They have no prescribed behavior at the boundary ∂F^e of the region of their orthogonality F^e, unlike the orbit functions which are either symmetric or antisymmetric with respect to their boundary ∂F. The region F^e is a union of two adjacent copies of the region F. Discretizations of the functions over lattice fragments $L \cap F$ and $L \cap F^e$, particularly their orthogonality over $L \cap F^e$, require different specific values for a number of constants required for groups G.

The purpose of the paper is to provide all of the information needed for the Fourier analysis of n-dimensional digital data in terms of their E-function expansions, in the context of an admissible symmetry group G. We suppose that G is one of the classical simple Lie groups with the Lie algebras of types A_n, B_n, C_n and D_n.

In section 2, some standard properties of simple Lie groups and/or simple Lie algebras are recalled. The properties of the group W^e not generally available elsewhere are important. In section 3, we describe the lattice grids $F^e_M \subset \mathbb{R}^n$, where the digital data are provided. The density of the grid is controlled by our choice of the integer M. For any grid F^e_M, there are only finitely many distinct E-functions that are orthogonal on the grid. Also, the lowest set of such functions is described. The functions are labeled by the grid of points Λ^e_M. In section 4, the properties of the E-functions are described for each point of Λ^e_M. Discrete E-transforms are presented in section 5. Concluding comments and remarks are provided in section 6.

2. Pertinent properties of simple Lie groups and their Lie algebras

2.1. Definitions and notations

Consider the Lie algebra of the compact simple Lie group G of rank n, with the set of simple roots $\Delta = \{\alpha_1, \dots, \alpha_n\}$, spanning the Euclidean space $\mathbb{R}^n$ [6–8].

By uniform and standard methods for G of any type and rank, a number of related quantities and virtually all the properties of G are determined from Δ. We make use of the following.

- The highest root $\xi \equiv -\alpha_0 = m_1\alpha_1 + \cdots + m_n\alpha_n$. Here, the coefficients m_j are known positive integers, also called the marks of G.
- The Coxeter number $m = 1 + m_1 + \cdots + m_n$ of G.
- The Cartan matrix C

$$C_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}, \qquad i, j \in \{1, \dots, n\}.$$

- The order c of the center of G

$$c = \det C. \tag{1}$$

- The dual weight lattice

$$P^\vee = \{\omega^\vee \in \mathbb{R}^n \mid \langle \omega^\vee, \alpha\rangle \in \mathbb{Z}, \forall \alpha \in \Delta\} = \mathbb{Z}\omega_1^\vee + \cdots + \mathbb{Z}\omega_n^\vee.$$

- The root lattice Q of G

$$Q = \{\alpha \in \mathbb{R}^n | \langle \alpha, \omega^\vee \rangle \in \mathbb{Z}, \ \forall \omega^\vee \in P^\vee\} = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n \,. \tag{2}$$

- The dual root lattice

$$Q^\vee = \mathbb{Z}\alpha_1^\vee + \cdots + \mathbb{Z}\alpha_n^\vee, \qquad \text{where} \quad \alpha_i^\vee = \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}.$$

2.2. Weyl group and affine Weyl group

The properties of Weyl groups and affine Weyl groups can be found for example in [9, 10]. The finite Weyl group W is generated by n reflections r_α, $\alpha \in \Delta$, in $(n-1)$-dimensional 'mirrors' orthogonal to simple roots intersecting at the origin:

$$r_{\alpha_i} a \equiv r_i a = a - \frac{2\langle a, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \alpha_i, \qquad a \in \mathbb{R}^n.$$

The infinite affine Weyl group W^{aff} is the semidirect product of the Abelian group of translations $Q^\vee$ and of the Weyl group W:

$$W^{\mathrm{aff}} = Q^\vee \rtimes W. \tag{3}$$

Equivalently, W^{aff} is generated by reflections r_i and reflection r_0, where

$$r_0 a = r_\xi a + \frac{2\xi}{\langle \xi, \xi \rangle}, \qquad r_\xi a = a - \frac{2\langle a, \xi \rangle}{\langle \xi, \xi \rangle} \xi, \qquad a \in \mathbb{R}^n.$$

The fundamental region F of W^{aff} is the convex hull of the points $\left\{0, \frac{\omega_1^\vee}{m_1}, \ldots, \frac{\omega_n^\vee}{m_n}\right\}$:

$$\begin{aligned} F &= \left\{ y_1 \omega_1^\vee + \cdots + y_n \omega_n^\vee | y_0, \ldots, y_n \in \mathbb{R}^{\geqslant 0}, \ y_0 + y_1 m_1 + \cdots + y_n m_n = 1 \right\} \\ &= \left\{ a \in \mathbb{R}^n | \langle a, \alpha \rangle \geqslant 0, \forall \alpha \in \Delta, \ \langle a, \xi \rangle \leqslant 1 \right\}. \end{aligned} \tag{4}$$

Since F is a fundamental region of W^{aff}, we have the following.

(1) For any $a \in \mathbb{R}^n$ there exists $a' \in F$, $w \in W$ and $q^\vee \in Q^\vee$ such that

$$a = wa' + q^\vee. \tag{5}$$

(2) If $a, a' \in F$ and $a' = w^{\mathrm{aff}} a$, $w^{\mathrm{aff}} \in W^{\mathrm{aff}}$, then $a = a'$, i.e. if there exist $w \in W$ and $q^\vee \in Q^\vee$ such that $a' = wa + q^\vee$, then

$$a' = a = wa + q^\vee. \tag{6}$$

(3) Consider a point $a = y_1 \omega_1^\vee + \cdots + y_n \omega_n^\vee \in F$, such that $y_0 + y_1 m_1 + \cdots + y_n m_n = 1$. The isotropy group

$$\mathrm{Stab}_{W^{\mathrm{aff}}}(a) = \{w^{\mathrm{aff}} \in W^{\mathrm{aff}} \mid w^{\mathrm{aff}} a = a\} \tag{7}$$

of the point a is trivial, $\mathrm{Stab}_{W^{\mathrm{aff}}}(a) = 1$, if $a \in \mathrm{int}(F)$, where $\mathrm{int}(F)$ denotes the interior of F, i.e. all $y_i > 0$, $i = 0, \ldots, n$. Otherwise the group $\mathrm{Stab}_{W^{\mathrm{aff}}}(a)$ is finite and generated by such r_i for which $y_i = 0$, $i = 0, \ldots, n$.

2.3. Even Weyl group and even affine Weyl group

Elements of the Weyl group W are orthogonal linear transformations of the space $\mathbb{R}^n$. A subgroup of W of the elements $w \in W$ with determinant $\det w = 1$ is called the even Weyl group W^e, i.e.

$$W^e = \{w \in W \,|\, \det w = 1\}.$$

The subgroup W^e can be viewed as the kernel of the homomorphism $\det : W \ni w \mapsto \det w$. Since $\ker \det = W^e$ and $\det(W) = \{\pm 1\}$, group W^e is a normal subgroup of W and

$$|W| = 2|W^e|.$$

The infinite even affine Weyl group W_e^{aff} is the semidirect product of the group of translations $Q^\vee$ and of the even Weyl group W^e:

$$W_e^{\mathrm{aff}} = Q^\vee \rtimes W^e. \tag{8}$$

We choose some fixed $j \in \{1, \dots, n\}$ and define the set F^e by

$$F^e = F \cup r_j \mathrm{int}(F). \tag{9}$$

Note that F^e consists of two disjoint parts: the closed simplex F and the open interior of the simplex $r_j \mathrm{int}(F)$. From this decomposition, we also obtain the formula for the volume of F^e:

$$\mathrm{vol}(F^e) = 2\mathrm{vol}(F).$$

In the following proposition, we show that F^e is a fundamental region of the even affine Weyl group W_e^{aff}.

Proposition 2.1. *For the set F^e, the following holds.*

(1) For any $a \in \mathbb{R}^n$ there exist $a' \in F^e$, $w \in W^e$ and $q^\vee \in Q^\vee$, such that

$$a = wa' + q^\vee. \tag{10}$$

(2) If $a, a' \in F^e$ and $a' = w^{\mathrm{aff}} a$, $w^{\mathrm{aff}} \in W_e^{\mathrm{aff}}$, then $a = a'$, i.e. if there exist $w \in W^e$ and $q^\vee \in Q^\vee$ such that $a' = wa + q^\vee$, then

$$a' = a = wa + q^\vee. \tag{11}$$

(3) Consider a point $a \in F^e$. If $a \in \mathrm{int}(F)$ or $a \in r_j \mathrm{int}(F)$, then the isotropy group

$$\mathrm{Stab}_{W_e^{\mathrm{aff}}}(a) = \left\{ w^{\mathrm{aff}} \in W_e^{\mathrm{aff}} \,\middle|\, w^{\mathrm{aff}} a = a \right\} \tag{12}$$

is trivial, $\mathrm{Stab}_{W_e^{\mathrm{aff}}}(a) = 1$. If $a \in F \backslash \mathrm{int}(F)$, then it holds that

$$|\mathrm{Stab}_{W^{\mathrm{aff}}}(a)| = 2 \left| \mathrm{Stab}_{W_e^{\mathrm{aff}}}(a) \right|. \tag{13}$$

Proof.

(1) Suppose we have some $a \in \mathbb{R}^n$. It follows from (5) that there exist $a' \in F$, $w \in W$ and $q^\vee \in Q^\vee$ such that $a = wa' + q^\vee$. If $\det w = 1$, then we find $a' \in F \subset F^e$, $w \in W^e$ and $q^\vee \in Q^\vee$ that satisfy (5). Suppose that $\det w = -1$ and

   (1) $a' \in \mathrm{int}(F)$. Since $F^e \equiv F \cup r_j \mathrm{int}(F)$ we have $r_j a' \in F^e$. Taking into account that $\det r_j = -1$ and $r_j^2 = 1$ we obtain $wr_j \in W^e$ and $a = (wr_j) r_j a' + q^\vee$.

   (2) $a' \in F \backslash \mathrm{int}(F)$. We have from (7) that the stabilizer $\mathrm{Stab}_{W^{\mathrm{aff}}}(a')$ is non-trivial and contains some r_i, $i \in \{0, \dots, n\}$ such that $r_i a' = a'$. If $i \in \{1, \dots, n\}$, then we have $wr_i \in W^e$ and $a = (wr_i) a' + q^\vee$. If $i = 0$, then we have $wr_\xi \in W^e$ and $a = (wr_\xi) a' + q'^\vee$ where $q'^\vee = 2w\xi / \langle \xi, \xi \rangle + q^\vee \in Q^\vee$.

(2) Suppose we have $a, a' \in F^e$ and $w \in W^e$, $q \in Q^\vee$ such that

$$a' = wa + q^\vee. \tag{14}$$

Since F^e consists of two disjoint parts F and $r_j \mathrm{int}(F)$, we distinguish the following cases.

(1) $a, a' \in F$. It follows immediately from (6) that $a = a'$.

(2) $a, a' \in r_j \mathrm{int}(F)$. Consider $b, b' \in \mathrm{int}(F)$ such that $a = r_j b$ and $a' = r_j b'$. Then $b' = r_j w r_j b + r_j q^\vee$ and from (6) we obtain $b = b'$, i.e. $a = a'$.

(3) $a' \in F$, $a \in r_j \mathrm{int}(F)$. Consider $a = r_j b$, $b \in \mathrm{int}(F)$. Then $a' = w r_j b + q^\vee$ and from (6) we have that $a' = b$. Since the stabilizer of the point $b \in \mathrm{int}(F)$ is trivial, $\mathrm{Stab}_{W^{\mathrm{aff}}}(b) = 1$, we obtain $w r_j = 1$. We conclude that $\det w = -1$, which is contradictory to the assumption $w \in W^e$ in (14) and thus, this case cannot occur.

(3) If $a \in \mathrm{int}(F)$, then from (7) we have that the stabilizer $\mathrm{Stab}_{W^{\mathrm{aff}}}(a)$ is trivial. Since the stabilizers of the points a and r_j are conjugated, the stabilizer $\mathrm{Stab}_{W^{\mathrm{aff}}}(r_j a)$ is also trivial. Then since $\mathrm{Stab}_{W_e^{\mathrm{aff}}}(a) \subset \mathrm{Stab}_{W^{\mathrm{aff}}}(a)$, we have $\mathrm{Stab}_{W_e^{\mathrm{aff}}}(a) = \mathrm{Stab}_{W_e^{\mathrm{aff}}}(r_j a) = 1$.

We have from (3) that for any $w^{\mathrm{aff}} \in W^{\mathrm{aff}}$ there exist a unique $w \in W$ and a unique shift $T(q^\vee)$ such that $w^{\mathrm{aff}} = T(q^\vee)w$. Define a homomorphism $\tau : \mathrm{Stab}_{W^{\mathrm{aff}}}(a) \to \{\pm 1\}$ for $w^{\mathrm{aff}} \in \mathrm{Stab}_{W^{\mathrm{aff}}}(a)$ by $\tau(w^{\mathrm{aff}}) = \tau(T(q^\vee)w) = \det w$. If $a \in F \backslash \mathrm{int}(F)$, then we have from (7) that the stabilizer $\mathrm{Stab}_{W^{\mathrm{aff}}}(a)$ is non-trivial, finite and contains some r_i, $i \in \{0, \ldots, n\}$. Since $\tau(r_i) = -1$, $i \in \{0, \ldots, n\}$ holds, we have $\tau(\mathrm{Stab}_{W^{\mathrm{aff}}}(a)) = \{\pm 1\}$. Then since $\ker \tau = \mathrm{Stab}_{W_e^{\mathrm{aff}}}(a)$, we conclude that $\mathrm{Stab}_{W^{\mathrm{aff}}}(a)/\mathrm{Stab}_{W_e^{\mathrm{aff}}}(a) \cong \{\pm 1\}$. □

2.4. *Action of W^e on the maximal torus $\mathbb{R}^n/Q^\vee$*

If we have two elements $a, a' \in \mathbb{R}^n$ such that $a' - a = q^\vee$, with $q^\vee \in Q^\vee$, then for $w \in W^e$ we have $wa - wa' = wq^\vee \in Q^\vee$, i.e. we have a natural action of W^e on the torus $\mathbb{R}^n/Q^\vee$. For $x \in \mathbb{R}^n/Q^\vee$ we denote the isotropy group and its order by

$$h_x^e \equiv |\mathrm{Stab}^{\mathrm{e}}(x)|, \qquad \mathrm{Stab}^{\mathrm{e}}(x) = \{w \in W^e | wx = x\}. \tag{15}$$

We denote the orbit and its order by

$$\varepsilon^e(x) \equiv |W^e x|, \qquad W^e x = \{wx \in \mathbb{R}^n/Q^\vee | w \in W^e\}.$$

Clearly, we have

$$\varepsilon^e(x) = \frac{|W^e|}{h_x^e}. \tag{16}$$

Proposition 2.2.

(1) For any $x \in \mathbb{R}^n/Q^\vee$, there exist $x' \in F^e \cap \mathbb{R}^n/Q^\vee$ and $w \in W^e$ such that

$$x = wx'. \tag{17}$$

(2) If $x, x' \in F^e \cap \mathbb{R}^n/Q^\vee$ and $x' = wx$, $w \in W^e$, then

$$x' = x = wx. \tag{18}$$

(3) If $x \in F^e \cap \mathbb{R}^n/Q^\vee$, i.e. $x = a + Q^\vee$, $a \in F^e$, then

$$\mathrm{Stab}^{\mathrm{e}}(x) \cong \mathrm{Stab}_{W_e^{\mathrm{aff}}}(a). \tag{19}$$

Proof.

(1) Follows directly from (10).
(2) Follows directly from (11).

(3) We have from (8) that for any $w^{\mathrm{aff}} \in W_e^{\mathrm{aff}}$ there exist a unique $w \in W^e$ and a unique shift $T(q^\vee)$ such that $w^{\mathrm{aff}} = T(q^\vee)w$. Define a homomorphism $\psi : \mathrm{Stab}_{W_e^{\mathrm{aff}}}(a) \to W^e$ for $w^{\mathrm{aff}} \in \mathrm{Stab}_{W_e^{\mathrm{aff}}}(a)$ by $\psi(w^{\mathrm{aff}}) = \psi(T(q^\vee)w) = w$. If $a = w^{\mathrm{aff}}a = wa + q^\vee$, then $a - wa = q^\vee \in Q^\vee$, i.e. we obtain $w \in \mathrm{Stab}^{\mathrm{e}}(x)$ and vice versa. Thus, $\psi(\mathrm{Stab}_{W_e^{\mathrm{aff}}}(a)) = \mathrm{Stab}^{\mathrm{e}}(x)$ holds. We also have

$$\ker\psi = \left\{T(q^\vee) \in \mathrm{Stab}_{W_e^{\mathrm{aff}}}(a)\right\} = 1.$$

□

2.5. Dual Lie algebra

The set of simple dual roots $\Delta^\vee = \left\{\alpha_1^\vee, \ldots, \alpha_n^\vee\right\}$ is a system of simple roots of some simple Lie algebra. The system $\Delta^\vee$ also spans the Euclidean space $\mathbb{R}^n$.

The dual system $\Delta^\vee$ determines the following.

- The highest dual root $\eta \equiv -\alpha_0^\vee = m_1^\vee\alpha_1^\vee + \cdots + m_n^\vee\alpha_n^\vee$. Here, the coefficients $m_j^\vee$ are called the dual marks of G.
- The dual Cartan matrix $C^\vee$
$$C_{ij}^\vee = \frac{2\langle\alpha_i^\vee, \alpha_j^\vee\rangle}{\langle\alpha_j^\vee, \alpha_j^\vee\rangle} = C_{ji}, \qquad i, j \in \{1, \ldots, n\}.$$
- The dual root lattice
$$Q^\vee = \mathbb{Z}\alpha_1^\vee + \cdots + \mathbb{Z}\alpha_n^\vee.$$
- The root lattice
$$Q = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n, \qquad \text{where} \quad \alpha_i = \frac{2\alpha_i^\vee}{\langle\alpha_i^\vee, \alpha_i^\vee\rangle}.$$
- The $\mathbb{Z}$-dual lattice
$$P = \{\omega \in \mathbb{R}^n | \langle\omega, \alpha^\vee\rangle \in \mathbb{Z}, \forall\alpha^\vee \in \Delta^\vee\} = \mathbb{Z}\omega_1 + \cdots + \mathbb{Z}\omega_n.$$

2.6. Dual affine Weyl group and its even subgroup

Dual affine Weyl group $\widehat{W}^{\mathrm{aff}}$ is a semidirect product of the group of shifts Q and the Weyl group W:

$$\widehat{W}^{\mathrm{aff}} = Q \rtimes W. \tag{20}$$

Equivalently, $\widehat{W}^{\mathrm{aff}}$ is generated by reflections r_i and $r_0^\vee$, where

$$r_0^\vee a = r_\eta a + \frac{2\eta}{\langle\eta, \eta\rangle}, \qquad r_\eta a = a - \frac{2\langle a, \eta\rangle}{\langle\eta, \eta\rangle}\eta, \qquad a \in \mathbb{R}^n.$$

The fundamental region $F^\vee$ of $\widehat{W}^{\mathrm{aff}}$ is the convex hull of the vertices $\left\{0, \frac{\omega_1}{m_1^\vee}, \ldots, \frac{\omega_n}{m_n^\vee}\right\}$:

$$\begin{aligned} F^\vee &= \left\{z_1\omega_1 + \cdots + z_n\omega_n \,\middle|\, z_0, \ldots, z_n \in \mathbb{R}^{\geqslant 0}, z_0 + z_1m_1^\vee + \cdots + z_nm_n^\vee = 1\right\} \\ &= \left\{a \in \mathbb{R}^n \mid \langle a, \alpha^\vee\rangle \geqslant 0, \forall\alpha^\vee \in \Delta^\vee, \langle a, \eta\rangle \leqslant 1\right\}. \end{aligned} \tag{21}$$

The dual even affine Weyl group $\widehat{W}_e^{\mathrm{aff}}$ is the semidirect product of the group of translations Q, and of the even Weyl group W^e:

$$\widehat{W}_e^{\mathrm{aff}} = Q \rtimes W^e. \tag{22}$$

We choose some fixed $j \in \{1, \ldots, n\}$ and define the set $F^{e\vee}$ by

$$F^{e\vee} = F^\vee \cup r_j\mathrm{int}(F^\vee). \tag{23}$$

Analogously to proposition 2.1, we obtain that $F^{e\vee}$ is a fundamental region of the dual even affine Weyl group $\widehat{W}_e^{\mathrm{aff}}$.

Proposition 2.3. *For the set $F^{e\vee}$, the following holds.*

(1) For any $a \in \mathbb{R}^n$, there exist $a' \in F^{e\vee}$, $w \in W^e$ and $q \in Q$ such that

$$a = wa' + q. \tag{24}$$

(2) If $a, a' \in F^{e\vee}$ and $a' = w^{\mathrm{aff}}a$, $w^{\mathrm{aff}} \in \widehat{W}_e^{\mathrm{aff}}$, then $a = a'$, i.e. if there exist $w \in W^e$ and $q \in Q$ such that $a' = wa + q$, then

$$a' = a = wa + q. \tag{25}$$

(3) Consider a point $a \in F^{e\vee}$. If $a \in \mathrm{int}(F^\vee)$ or $a \in r_j\mathrm{int}(F^\vee)$, then the isotropy group

$$\mathrm{Stab}_{\widehat{W}_e^{\mathrm{aff}}}(a) = \left\{w^{\mathrm{aff}} \in \widehat{W}_e^{\mathrm{aff}} \mid w^{\mathrm{aff}}a = a\right\} \tag{26}$$

is trivial, $\mathrm{Stab}_{\widehat{W}_e^{\mathrm{aff}}}(a) = 1$. *If $a \in F^\vee \backslash \mathrm{int}(F^\vee)$, then it holds*

$$|\mathrm{Stab}_{\widehat{W}^{\mathrm{aff}}}(a)| = 2\left|\mathrm{Stab}_{\widehat{W}_e^{\mathrm{aff}}}(a)\right|, \tag{27}$$

where

$$\mathrm{Stab}_{\widehat{W}^{\mathrm{aff}}}(a) = \{w^{\mathrm{aff}} \in \widehat{W}^{\mathrm{aff}} \mid w^{\mathrm{aff}}a = a\}. \tag{28}$$

3. Grids F_M^e and Λ_M^e

3.1. Grid F_M^e

The grid F_M^e is the finite fragment of the lattice $\frac{1}{M}P^\vee$ which is found inside of F^e. Suppose we have a fixed $M \in \mathbb{N}$ and consider the W-invariant group $\frac{1}{M}P^\vee/Q^\vee$. The group $\frac{1}{M}P^\vee/Q^\vee$ is finite with the order

$$\left|\frac{1}{M}P^\vee/Q^\vee\right| = cM^n. \tag{29}$$

We define the grid F_M^e as such cosets from $\frac{1}{M}P^\vee/Q^\vee$ which have a representative element in the fundamental domain F^e:

$$F_M^e \equiv \frac{1}{M}P^\vee/Q^\vee \cap F^e.$$

From relation (17), we have that

$$W^eF_M^e = \frac{1}{M}P^\vee/Q^\vee. \tag{30}$$

The grid F_M^e can be viewed as a union of two disjoint grids—the grid $F_M \equiv \frac{1}{M}P^\vee/Q^\vee \cap F$ and the reflection r_j of its interior $\widetilde{F}_M \equiv \frac{1}{M}P^\vee/Q^\vee \cap \mathrm{int}(F)$:

$$F_M^e = F_M \cup r_j\widetilde{F}_M. \tag{31}$$

We obtain from (4) that the set F_M, or more precisely its representative points, can be identified as

$$F_M = \left\{\frac{s_1}{M}\omega_1^\vee + \cdots + \frac{s_n}{M}\omega_n^\vee \;\middle|\; s_0, s_1, \ldots, s_n \in \mathbb{Z}^{\geqslant 0},\ s_0 + \sum_{i=1}^n s_i m_i = M\right\}. \tag{32}$$

The reflection r_j of its interior $\widetilde{F}_M$ is given by

$$r_j\widetilde{F}_M = \left\{\frac{s_1'}{M}\omega_1^\vee + \cdots + \frac{s_j'}{M}(\omega_j^\vee - \alpha_j^\vee) + \cdots + \frac{s_n'}{M}\omega_n^\vee \;\middle|\; s_0', s_1', \ldots, s_n' \in \mathbb{N},\ s_0' + \sum_{i=1}^n s_i'm_i = M\right\}. \tag{33}$$

3.2. Number of elements of F^e_M

The number of elements of F^e_M could be obtained by combining results from [1].

Proposition 3.1. *Let m be the Coxeter number. Then*

$$\left|F^e_M\right| = \begin{cases} |F_M| & M < m \\ |F_M| + 1 & M = m \\ |F_M| + |F_{M-m}| & M > m. \end{cases}$$

Proof. Considering the equality $|\widetilde{F}_M| = |r_j \widetilde{F}_M|$, we obtain from the disjoint decomposition (31) that

$$\left|F^e_M\right| = |F_M| + |\widetilde{F}_M|. \tag{34}$$

It was shown in proposition 3.5 in [1] that

$$|\widetilde{F}_M| = \begin{cases} 0 & M < m \\ 1 & M = m \\ |F_{M-m}| & M > m. \end{cases} \tag{35}$$

□

Theorem 3.2. *The numbers of points of the grid F^e_M of Lie algebras A_n, B_n, C_n, D_n are given by the following relations.*

(1) $A_n,\ n \geqslant 1$,

$$\left|F^e_M(A_n)\right| = \binom{n+M}{n} + \binom{M-1}{n}$$

(2) $C_n,\ n \geqslant 2$,

$$\left|F^e_{2k}(C_n)\right| = \binom{n+k}{n} + \binom{n+k-1}{n} + \binom{k}{n} + \binom{k-1}{n}$$

$$\left|F^e_{2k+1}(C_n)\right| = 2\binom{n+k}{n} + 2\binom{k}{n}$$

(3) $B_n,\ n \geqslant 3$,

$$\left|F^e_M(B_n)\right| = \left|F^e_M(C_n)\right|$$

(4) $D_n,\ n \geqslant 4$,

$$\left|F^e_{2k}(D_n)\right| = \binom{n+k}{n} + 6\binom{n+k-1}{n} + \binom{n+k-2}{n} + \binom{k+1}{n} + 6\binom{k}{n} + \binom{k-1}{n}$$

$$\left|F^e_{2k+1}(D_n)\right| = 4\binom{n+k}{n} + 4\binom{n+k-1}{n} + 4\binom{k+1}{n} + 4\binom{k}{n}.$$

Proof. For the case A_n, we have from [1] that $m = n + 1$ and $|F_M(A_n)| = \binom{n+M}{n}$. It can be verified directly that the formula

$$|\widetilde{F}_M(A_n)| = \binom{M-1}{n}$$

satisfies (35) for all values of $M \in \mathbb{N}$. The result follows from (34). Analogously we obtain formulas for algebras B_n, C_n and D_n.

□

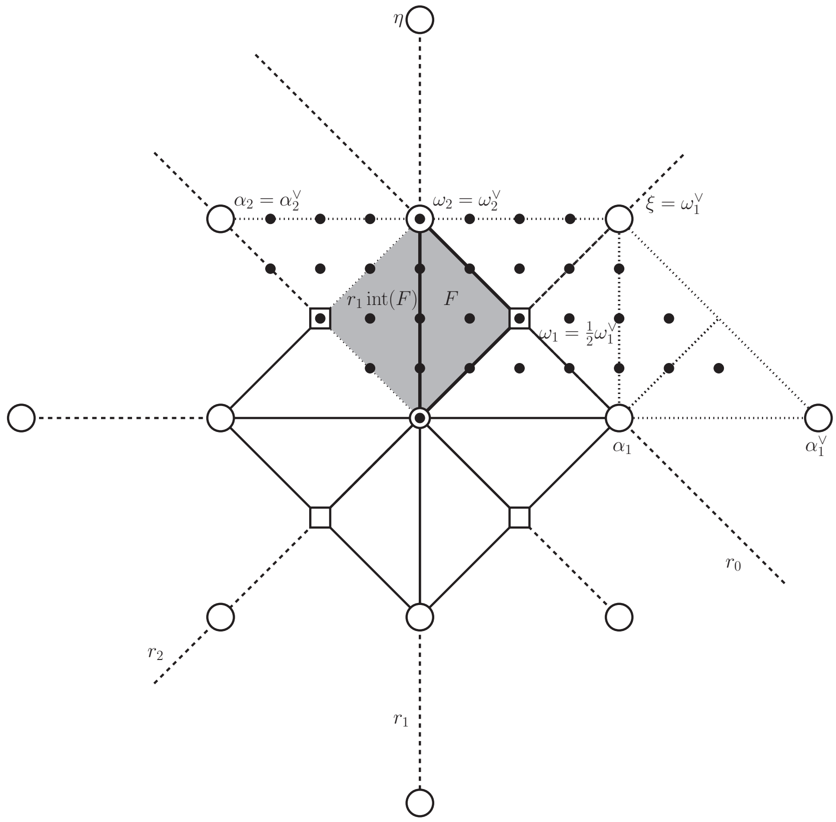

Figure 1. Coset representants of $\frac{1}{4}P^\vee/Q^\vee$ of C_2; coset representants are shown as 32 black dots, the gray area is the fundamental domain $F^e = F \cup r_1\mathrm{int}(F)$ containing ten points of $F_4^e(C_2)$. Dashed lines represent 'mirrors' r_0, r_1 and r_2. Circles are elements of the root lattice Q, together with squares they are elements of the weight lattice P.

Using explicit formulas for $|F_M|$ from [1], the number $\left|F_M^e\right|$ of the five exceptional Lie algebras can be obtained similarly from proposition 3.1.

Example 3.1. For the Lie algebra C_2, we have the Coxeter number $m = 4$ and $c = 2$. Consider for example $M = 4$. For the order of the group $\frac{1}{4}P^\vee/Q^\vee$, we have from (29) that $\left|\frac{1}{4}P^\vee/Q^\vee\right| = 32$ and according to theorem 3.2 we calculate $\left|F_4^e(C_2)\right| = 10$. The coset representants of $\frac{1}{4}P^\vee/Q^\vee$ and the fundamental domain F^e are depicted in figure 1.

3.3. Grid Λ_M^e

The points of Λ_M^e are the weights that specify E-functions belonging to the same pairwise orthogonal set. Further on, we consider E-functions that are sampled on the points F_M^e. We consider the lowest possible set of such points. The number of points of Λ_M^e coincides with the number of points of F_M^e.

The W-invariant group P/MQ is isomorphic to the group $\frac{1}{M}P^\vee/Q^\vee$ and its order is given by

$$|P/MQ| = cM^n.$$

Define the grid Λ^e_M as such cosets from P/MQ with representative elements in $MF^{e\vee}$:

$$\Lambda^e_M \equiv MF^{e\vee} \cap P/MQ.$$

The grid Λ^e_M can be viewed as a union of two disjoint grids—the grid $\Lambda_M \equiv P/MQ \cap MF^\vee$ and the reflection r_j of its interior $\widetilde{\Lambda}_M \equiv P/MQ \cap \mathrm{int}(MF^\vee)$:

$$\Lambda^e_M = \Lambda_M \cup r_j\widetilde{\Lambda}_M. \tag{36}$$

We have from (21) that the set Λ_M, or more precisely its representative points, can be identified as

$$\Lambda_M = \left\{ t_1\omega_1 + \cdots + t_n\omega_n \,\middle|\, t_0, t_1, \ldots, t_n \in \mathbb{Z}^{\geqslant 0},\ t_0 + \sum_{i=1}^n t_i m_i^\vee = M \right\}. \tag{37}$$

The reflection r_j of its interior is given by

$$r_j\widetilde{\Lambda}_M = \left\{ t'_1\omega_1 + \cdots + t'_j(\omega_j - \alpha_j) + \cdots + t'_n\omega_n \,\middle|\, t'_0, t'_1, \ldots, t'_n \in \mathbb{N},\ t'_0 + \sum_{i=1}^n t'_i m_i^\vee = M \right\}. \tag{38}$$

Since the n-tuple of dual marks $\left(m_1^\vee, \ldots, m_n^\vee\right)$ is a certain permutation of the n-tuple $(m_1, \ldots, m_n)$, we have from (32) and (37) that $|F_M| = |\Lambda_M|$, and from (33) and (38) that $|r_j\widetilde{\Lambda}_M| = |r_j\widetilde{F}_M|$. Taking into account disjoint decompositions (31) and (36), we conclude that

$$\left|F^e_M\right| = \left|\Lambda^e_M\right|. \tag{39}$$

3.4. *Action of* W^e *on* P/MQ

If we have two elements $b, b' \in \mathbb{R}^n$ such that $b' - b = Mq$, with $q \in Q$, then for $w \in W^e$ we have $wb - wb' = wMq \in MQ$, i.e. we have a natural action of W^e on the quotient group $\mathbb{R}^n/MQ$. For $\lambda \in \mathbb{R}^n/MQ$ we denote the order of the stabilizer

$$h^{e\vee}_\lambda \equiv \left|\mathrm{Stab}^\vee_e(\lambda)\right|, \qquad \mathrm{Stab}^\vee_e(\lambda) = \{w \in W^e | w\lambda = \lambda\}. \tag{40}$$

Proposition 3.3.

(1) For any $\lambda \in P/MQ$ *there exists* $\lambda' \in \Lambda^e_M$ *and* $w \in W^e$ *such that*

$$\lambda = w\lambda'. \tag{41}$$

(2) If $\lambda, \lambda' \in \Lambda^e_M$ *and* $\lambda' = w\lambda$, $w \in W^e$, *then*

$$\lambda' = \lambda = w\lambda. \tag{42}$$

(3) If $\lambda \in MF^{e\vee} \cap \mathbb{R}^n/MQ$, *i.e.* $\lambda = b + MQ$, $b \in MF^{e\vee}$, *then*

$$\mathrm{Stab}^\vee_e(\lambda) \cong \mathrm{Stab}_{\widehat{W}^{\mathrm{aff}}_e}(b/M). \tag{43}$$

Proof.

(1) Let $\lambda \in P/MQ$ be of the form $\lambda = p + MQ$, $p \in P$. From (24) it follows that there exist $p' \in F^{e\vee}$, $w \in W^e$ and $q \in Q$ such that

$$\frac{1}{M}p = wp' + q,$$

i.e. $p = wMp' + Mq$. From W-invariance of P we have that $Mp' \in P$, the class $\lambda' = Mp' + MQ$ is from Λ^e_M and (41) holds.

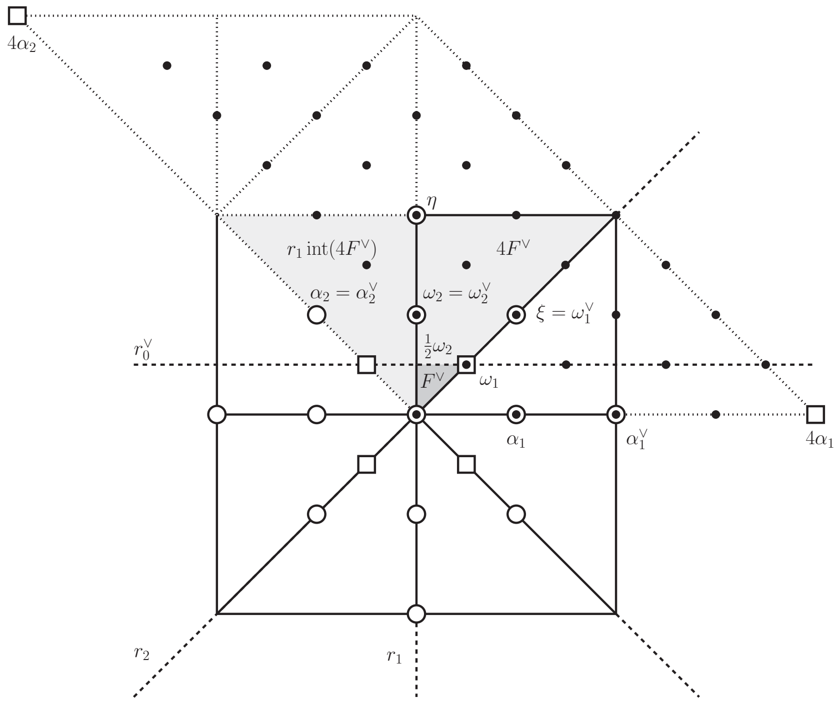

Figure 2. The cosets representants of $P/4Q$ of C_2; the cosets representants are shown as 32 black dots, the darker gray area is the fundamental domain $F^\vee$, the lighter gray area is the domain $4F^{e\vee} = 4F^\vee \cup r_1\mathrm{int}(4F^\vee)$ which contains ten elements of $\Lambda_4^e(C_2)$. The dashed lines represent dual 'mirrors' $r_0^\vee, r_1, r_2$. The circles and squares coincide with those in figure 1.

(2) Let $\lambda, \lambda' \in P/MQ$ be of the form $\lambda = p + MQ$, $\lambda' = p' + MQ$ and $p, p' \in MF^{e\vee}$. Suppose that

$$p' = wp + Mq, \qquad q \in Q,\ w \in W^e.$$

Then $p/M, p'/M \in F^{e\vee}$ and it follows from (25) that $p = p'$.

(3) We have from (22) that for any $w^{\mathrm{aff}} \in \widehat{W}_e^{\mathrm{aff}}$ there exist unique $w \in W^e$ and unique shift $T(q)$ such that $w^{\mathrm{aff}} = T(q)w$. Define a homomorphism $\psi : \mathrm{Stab}_{\widehat{W}_e^{\mathrm{aff}}}(b/M) \to W^e$ for $w^{\mathrm{aff}} \in \mathrm{Stab}_{\widehat{W}_e^{\mathrm{aff}}}(b/M)$ by $\psi(w^{\mathrm{aff}}) = \psi(T(q)w) = w$. If $b/M = w^{\mathrm{aff}}(b/M) = w(b/M) + q$, then $b - wb = Mq \in MQ$, i.e. we obtain $w \in \mathrm{Stab}_e^\vee(\lambda)$ and vice versa. Thus, $\psi(\mathrm{Stab}_{\widehat{W}_e^{\mathrm{aff}}}(b/M)) = \mathrm{Stab}_e^\vee(\lambda)$ holds. We also have

$$\ker\psi = \left\{T(q) \in \mathrm{Stab}_{\widehat{W}_e^{\mathrm{aff}}}(b/M)\right\} = 1.$$

□

Example 3.2. For the Lie algebra C_2 we have $|P/4Q| = 32$ and according to (39) we have

$$\left|\Lambda_4^e(C_2)\right| = \left|F_4^e(C_2)\right| = 10.$$

The coset representants of $P/4Q$, the dual fundamental domain $F^{e\vee}$ and the grid $\Lambda_4^e(C_2) = 4F^{e\vee} \cap P/4Q$ are depicted in figure 2.

3.5. Calculation of h^e_x and $h^{e\vee}_\lambda$

Calculation procedures of $h_x \equiv |\mathrm{Stab}_{W^{\mathrm{aff}}}(x)|$ for any $x \in F_M$ and of $h^\vee_\lambda \equiv \left|\mathrm{Stab}_{\widehat{W}^{\mathrm{aff}}}(\lambda)\right|$, $\lambda \in \Lambda_M$ was derived in section 3.7 of [1]. These calculation procedures use extended Coxeter–Dynkin diagrams (DD) of G and their dual versions DD$^\vee$ (see e.g.[1]). Modifying these procedures by using relations (19), (13) and (43), (27), we deduce a calculation procedure for h^e_x, $h^{e\vee}_\lambda$, defined by (15), (40), for $x \in F^e_M$ and $\lambda \in \Lambda^e_M$.

Consider a point $x \in F^e_M = F_M \cup r_j \widetilde{F}_M$.

(1) If $x \in r_j \widetilde{F}_M$, then $h^e_x = 1$.
(2) Let $[s_0, \dots, s_n]$ be the corresponding coordinates of $x \in F_M$ from (32). If $s_0, \dots, s_n$ are all non-zero, then $h^e_x = 1$.
(3) If some of the coordinates $[s_0, \dots, s_n]$ are zero, then consider such a subgraph U of extended DD consisting only of those nodes i for which $s_i = 0$, $i = 0, \dots, n$. The subgraph U consists in general of several connected components U_l. Each component U_l is a (non-extended) DD of some compact simple Lie group G_l. Take corresponding orders of the Weyl groups $|W_l|$ of G_l. Then it holds

$$h^e_x = \frac{1}{2} \prod_l |W_l|.$$

We proceed similarly to determine $h^{e\vee}_\lambda$ when considering a point $\lambda \in \Lambda^e_M = \Lambda_M \cup r_j \widetilde{\Lambda}_M$.

(1) If $\lambda \in r_j \widetilde{\Lambda}_M$, then $h^{e\vee}_\lambda = 1$.
(2) Let $[t_0, \dots, t_n]$ be the corresponding coordinates of $\lambda \in \Lambda_M$ from (37). If $t_0, \dots, t_n$ are all non-zero, then $h^{e\vee}_\lambda = 1$.
(3) If some of the coordinates $[t_0, \dots, t_n]$ are zero, then consider such a subgraph U' of the extended DD$^\vee$ consisting only of those nodes i for which $t_i = 0$, $i = 0, \dots, n$. The subgraph U' consists in general of several connected components U'_l. Each component U'_l is a (non-extended) DD of some compact simple Lie group G'_l. Take corresponding orders of the Weyl groups $|W'_l|$ of G'_l. Then it holds

$$h^{e\vee}_\lambda = \frac{1}{2} \prod_l |W'_l|.$$

4. W^e-Invariant functions

The numbers h^e_x, $h^{e\vee}_\lambda$ and $\left|F^e_M\right|$, which were determined so far, are important for the properties of special functions, called E-functions when they are sampled on F^e_M. A detailed review of the properties of E-functions may be found in [5]. In this section the goal is to complete and make explicit the orthogonality properties of E-functions [11].

4.1. E-functions

We recall the definition of E-functions and show that they can be labeled by the finite set Λ^e_M when sampled on the grid F^e_M.

Consider $b \in P$ and recall that (normalized) E-functions can be defined as a mapping $\Xi_b : \mathbb{R}^n \to \mathbb{C}$:

$$\Xi_b(a) = \sum_{w \in W^e} \mathrm{e}^{2\pi \mathrm{i}\langle wb, a\rangle}. \tag{44}$$

The following properties of E-functions are crucial:

- symmetry with respect to $w \in W^e$

$$\Xi_b(wa) = \Xi_b(a) \tag{45}$$

$$\Xi_{wb}(a) = \Xi_b(a), \tag{46}$$

- invariance with respect to $q^\vee \in Q^\vee$

$$\Xi_b(a + q^\vee) = \Xi_b(a). \tag{47}$$

We investigate values of E-functions on the grid F^e_M. Suppose we have a fixed $M \in \mathbb{N}$ and $s \in \frac{1}{M} P^\vee$. From (47) it follows that we can consider Ξ_b as a function on classes $\frac{1}{M} P^\vee / Q^\vee$. From (17) and (45) it follows that we can consider Ξ_b only on the set $F^e_M \equiv \frac{1}{M} P^\vee / Q^\vee \cap F^e$. We also have

$$\Xi_{b+MQ}(s) = \Xi_b(s), \qquad s \in F^e_M$$

and thus we can consider the functions Ξ_λ on F^e_M parameterized by classes from $\lambda \in P/MQ$. Moreover, from (41) and (46) it follows that we can consider E-functions Ξ_λ on F^e_M parameterized by $\lambda \in \Lambda^e_M$ only.

5. Discrete orthogonality of *E*-functions

5.1. *Basic discrete orthogonality relations*

Discrete orthogonality of E-functions was discussed in general in [11]. Practical application of [11] is not completely straightforward. Therefore, we reformulate the basic facts and subsequently use them to make the discrete orthogonality over F^e_M described in detail. Basic orthogonality relations from [1, 11] are for any $\lambda, \lambda' \in P/MQ$ of the form

$$\sum_{y \in \frac{1}{M} P^\vee / Q^\vee} e^{2\pi i \langle \lambda - \lambda', y \rangle} = c M^n \delta_{\lambda,\lambda'}. \tag{48}$$

5.2. *Discrete orthogonality of E-functions*

We define the scalar product of two functions $f, g : F^e_M \to \mathbb{C}$ by

$$\langle f, g \rangle_{F^e_M} = \sum_{x \in F^e_M} \varepsilon^e(x) f(x) \overline{g(x)}, \tag{49}$$

where the numbers $\varepsilon^e(x)$ are determined by (16). We show that Λ^e_M, defined by (36), is the lowest maximal set of pairwise orthogonal E-functions.

Proposition 5.1. *For $\lambda, \lambda' \in \Lambda^e_M$ it holds*

$$\langle \Xi_\lambda, \Xi_{\lambda'} \rangle_{F^e_M} = c |W^e| M^n h^{e\vee}_\lambda \delta_{\lambda,\lambda'}, \tag{50}$$

where c, $h^{e\vee}_\lambda$, Ξ_λ were defined by (1), (40), (44), respectively, n is the rank of G.

Proof. The equality

$$\sum_{x \in F^e_M} \varepsilon^e(x) \Xi_\lambda(x) \overline{\Xi_{\lambda'}(x)} = \sum_{y \in \frac{1}{M} P^\vee / Q^\vee} \Xi_\lambda(y) \overline{\Xi_{\lambda'}(y)}$$

Table 1. The coefficients $\varepsilon^e(x)$ and $h^{e\vee}_\lambda$ of C_2. Assuming $s_0, s_1, s_2 > 0$, $t_0, t_1, t_2 > 0$.

$x \in F_M(C_2)$	$\varepsilon^e(x)$	$\lambda \in \Lambda_M(C_2)$	$h^{e\vee}_\lambda$
$[s_0, s_1, s_2]$	4	$[t_0, t_1, t_2]$	1
$[0, s_1, s_2]$	4	$[0, t_1, t_2]$	1
$[s_0, 0, s_2]$	4	$[t_0, 0, t_2]$	1
$[s_0, s_1, 0]$	4	$[t_0, t_1, 0]$	1
$[0, 0, s_2]$	1	$[0, 0, t_2]$	2
$[0, s_1, 0]$	2	$[0, t_1, 0]$	4
$[s_0, 0.0]$	1	$[t_0, 0, 0]$	4

follows from (18) and (30) and W^e-invariance of the expression $\Xi_\lambda(x)\overline{\Xi_{\lambda'}(x)}$. Then, using W^e-invariance of $\frac{1}{M}P^\vee/Q^\vee$ and (48), we have

$$\begin{aligned}\langle \Xi_\lambda, \Xi_{\lambda'}\rangle_{F^e_M} &= \sum_{w'\in W^e}\sum_{w\in W^e}\sum_{y\in \frac{1}{M}P^\vee/Q^\vee} \mathrm{e}^{2\pi\mathrm{i}\langle w\lambda - w'\lambda', y\rangle} = |W^e| \sum_{w'\in W^e}\sum_{y\in \frac{1}{M}P^\vee/Q^\vee} \mathrm{e}^{2\pi\mathrm{i}\langle \lambda - w'\lambda', y\rangle}\\ &= c|W^e|M^n \sum_{w'\in W^e} \delta_{w'\lambda',\lambda}.\end{aligned}$$

Since $\lambda, \lambda' \in \Lambda^e_M$ we have from (42) that

$$\sum_{w'\in W^e} \delta_{w'\lambda',\lambda} = h^{e\vee}_\lambda \delta_{\lambda,\lambda'}.$$

□

Example 5.1. The highest root ξ and the highest dual root η of C_2 are determined by the formulas

$$\xi = 2\alpha_1 + \alpha_2, \qquad \eta = \alpha_1^\vee + 2\alpha_2^\vee.$$

The even Weyl group of C_2 has four elements, $|W^e| = 4$, and we obtain for the determinant of the Cartan matrix $c = 2$. Thus, decomposition of the grid $F^e_M(C_2) = F_M(C_2) \cup r_1\widetilde{F}_M(C_2)$ is given by

$$\begin{aligned} F_M(C_2) &= \left\{ \frac{s_1}{M}\omega_1^\vee + \frac{s_2}{M}\omega_2^\vee \,\middle|\, s_0, s_1, s_2 \in \mathbb{Z}^{\geqslant 0}, s_0 + 2s_1 + s_2 = M \right\} \\ r_1\widetilde{F}_M(C_2) &= \left\{ \frac{-s_1'}{M}\omega_1^\vee + \frac{s_2' + 2s_1'}{M}\omega_2^\vee \,\middle|\, s_0', s_1', s_2' \in \mathbb{N}, s_0' + 2s_1' + s_2' = M \right\} \end{aligned} \tag{51}$$

and the grid of weights $\Lambda^e_M(C_2) = \Lambda_M(C_2) \cup r_1\widetilde{\Lambda}_M(C_2)$ is determined by

$$\begin{aligned} \Lambda_M(C_2) &= \{t_1\omega_1 + t_2\omega_2 \mid t_0, t_1, t_2 \in \mathbb{Z}^{\geqslant 0}, t_0 + t_1 + 2t_2 = M\} \\ r_1\widetilde{\Lambda}_M(C_2) &= \{-t_1'\omega_1 + (t_1' + t_2')\omega_2 \mid t_0', t_1', t_2' \in \mathbb{N}, t_0' + t_1' + 2t_2' = M\}. \end{aligned}$$

The discrete orthogonality relations of E-functions of C_2 which hold for any two functions Ξ_λ, $\Xi_{\lambda'}$ labeled by $\lambda, \lambda' \in \Lambda^e_M(C_2)$ are of the form (50). The coefficients $\varepsilon^e(x)$, $h^{e\vee}_\lambda$, which appear in (50), have according to section 3.5 the values $\varepsilon^e(x) = 4$, $h^{e\vee}_\lambda = 1$ for $x \in r_1\widetilde{F}_M(C_2)$, $\lambda \in r_1\widetilde{\Lambda}_M(C_2)$. We represent each point $x \in F_M(C_2)$ and each weight $\lambda \in \Lambda_M(C_2)$ by the coordinates $[s_0, s_1, s_2]$ and $[t_0, t_1, t_2]$ from relations (51) and (52), respectively. The values of the coefficients $\varepsilon^e(x)$, $h^{e\vee}_\lambda$ for $x \in F_M(C_2)$, $\lambda \in \Lambda_M(C_2)$ are listed in table 1.

5.3. Discrete E-transforms

Analogously to ordinary Fourier analysis, we define the interpolating functions Ξ^M:

$$\Xi^M(x) := \sum_{\lambda\in\Lambda^e_M} c_\lambda \Xi_\lambda(x), \qquad x \in \mathbb{R}^n \tag{53}$$

which are given in terms of the expansion functions Ξ_λ and expansion coefficients c_λ, whose values need to be determined. These interpolating functions can also be understood as finite cut-offs of infinite expansions.

Next we discretize equations (53). Suppose we have some function f sampled on the grid F^e_M. The interpolation of f consists of finding the coefficients c_λ in the interpolating functions (53) such that

$$\Xi^M(x) = f(x), \qquad x \in F^e_M. \tag{54}$$

Relations (39) and (50) allow the values $\Xi_\lambda(x)$ with $x \in F^e_M$, $\lambda \in \Lambda^e_M$ to be viewed as elements of a non-singular square matrix. This invertible matrix coincides with the matrix of the linear system (54). Thus, the coefficients c_λ can be uniquely determined. The formula for the calculation of c_λ, which is also called discrete E-transform, can be obtained by means of the calculation of standard Fourier coefficients:

$$c_\lambda = \frac{\langle f, \Xi_\lambda\rangle_{F^e_M}}{\langle \Xi_\lambda, \Xi_\lambda\rangle_{F^e_M}} = \left(c|W^e|M^n h^{e\vee}_\lambda\right)^{-1} \sum_{x\in F^e_M} \varepsilon^e(x) f(x)\overline{\Xi_\lambda(x)}. \tag{55}$$

We also have the corresponding Plancherel formula

$$\sum_{x\in F^e_M} \varepsilon^e(x)|f(x)|^2 = c|W^e|M^n \sum_{\lambda\in\Lambda^e_M} h^{e\vee}_\lambda\, |c_\lambda|^2 .$$

6. Concluding remarks

The E-functions of the paper have other undoubtedly useful properties that were not invoked here. Let us briefly mention six of them.

- Products of two E-functions with the same underlying Lie group and the same arguments $x \in \mathbb{R}^n$ but different subscripts, say λ and λ', decompose into the sum of E-functions. This is a powerful property that enables building of a set of recursion relations for constructing even larger E-functions.
- Points of the weight lattice P of G split into a few disjoint congruence classes. It is convenient to specify a congruence class of $\mu \in P$ by its congruence number [12]. There is a different linear function of μ for a different Lie group G. The weights of one W-orbit, hence also the weights of one W^e-orbit, belong to the same congruence class. As a consequence, all summands $e^{2\pi i\langle\mu,x\rangle}$ in an E-function have the weights μ from the same congruence class. Hence, the E-function has a well-defined congruence number. During multiplication of E-functions of the same G, the congruence numbers of the E-functions add up. In particular, the decomposition of a product of two E-functions must contain summands with the same congruence number. The number of congruence classes of a G is equal to the order of the center of the compact simple Lie group G.
- Another undoubtedly useful property of orbit functions, which plays no role in this paper, is the fact that they are eigenfunctions of the Laplace operator appropriate for the Lie group, and that the eigenvalues are known in all cases [5].

- The E-functions considered in the paper so far have the underlying simple group G. Suppose now that it is semisimple but not simple, say $G = G_1 \times G_2$, where G_1 and G_2 are simple. Then there are two options as to what to take for the even subgroup $W^e(G_1 \times G_2)$. The simpler of the two is to define $W^e(G_1 \times G_2) := W^e(G_1) \times W^e(G_2)$. The E-functions of $W^e(G_1) \times W^e(G_2)$ are then products of the E-functions of G_1 and G_2, etc, see [5]. Such an option is trivial, for example when $G = SU(2) \times SU(2)$.
  The option where $W^e(G_1 \times G_2)$ is bigger, namely the full even subgroup of $W(G_1) \times W(G_2)$, is somewhat more interesting. This option is still to be explored in the literature [13]. It is already non-trivial in the lowest case $G = SU(2) \times SU(2)$.
- The results presented in this paper are valid for any compact simple Lie group G, including the five exceptional cases. Explicit counting formulas for $|F^e_M|$ for these five cases, which were not provided here, can be straightforwardly put together using proposition 3.1 and the appendix in [1].
- The present work raises the question: under which conditions converge the series of the functions $\{\Xi^M\}_{M=1}^{\infty}$ assigned to a function $f : F^e \to \mathbb{C}$ by relations (53), (55).

Acknowledgments

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